

ALGEBRAIC RANK ON HYPERELLIPTIC GRAPHS AND GRAPHS OF GENUS 3

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ABSTRACT. Let $\bar{G} = (G, \omega)$ be a vertex-weighted graph, and δ a divisor class on G . Let $r_{\bar{G}}(\delta)$ denote the combinatorial rank of δ . Caporaso has introduced the algebraic rank $r_{\bar{G}}^{\text{alg}}(\delta)$ of δ , by using nodal curves with dual graph \bar{G} . In this paper, when \bar{G} is hyperelliptic or of genus 3, we show that $r_{\bar{G}}^{\text{alg}}(\delta) \geq r_{\bar{G}}(\delta)$ holds, generalizing our previous result. We also show that, with respect to the specialization map from a non-hyperelliptic curve of genus 3 to its reduction graph, any divisor on the graph lifts to a divisor on the curve of the same rank.

1. INTRODUCTION

Let k be an algebraically closed field. The correspondence between nodal curves over k and their (vertex-weighted) dual graphs appears naturally in algebraic geometry, as in the description of the stratification of the Deligne–Mumford moduli space of stable curves. Recently, a theory of divisors on graphs has been developed (see for example [2], [3], [5] and [6]). This enables one to study the relationship between *linear systems* on a nodal curve and *those* on the corresponding graph (and also between linear systems on the generic fiber and those on the dual graph of the special fiber of a semi-stable curve over a discrete valuation ring): See, for example, [1], [4], [7], [8], [9], [11] and [12]. In particular, a tropical proof of the Brill–Noether theorem has been obtained in [11].

In this development, Caporaso [9] has defined the algebraic rank $r_{\bar{G}}^{\text{alg}}(\delta)$ of a divisor class δ on a vertex-weighted graph $\bar{G} = (G, \omega)$, by using nodal curves with dual graph \bar{G} . It was shown in [9, Summary 3.4] that, on some simple graphs \bar{G} , the algebraic rank $r_{\bar{G}}^{\text{alg}}(\delta)$ equals the combinatorial rank $r_{\bar{G}}(\delta)$ for any divisor class δ . Further, Caporaso, Len and Melo in [10] have recently shown that $r_{\bar{G}}^{\text{alg}}(\delta) \leq r_{\bar{G}}(\delta)$ holds for any divisor class δ on any vertex-weighted graph \bar{G} . In [12, Proposition 1.5], we showed that, if $\text{char}(k) \neq 2$ and \bar{G} is a hyperelliptic vertex-weighted graph satisfying a certain assumption on the bridges of G , then $r_{\bar{G}}^{\text{alg}}(\delta) \geq r_{\bar{G}}(\delta)$ holds for any divisor class δ .

In this paper, firstly, we show, based on [12, Proposition 1.5], that the above assumption on the bridges for hyperelliptic graphs is not necessary.

Theorem 1.1. *Assume that $\text{char}(k) \neq 2$. Let $\bar{G} = (G, \omega)$ be a hyperelliptic vertex-weighted graph. Then, for any divisor class δ on G , we have $r_{\bar{G}}^{\text{alg}}(\delta) \geq r_{\bar{G}}(\delta)$.*

Secondly, we show the same inequality on non-hyperelliptic graphs of genus 3.

Theorem 1.2. *Let $\bar{G} = (G, \omega)$ be a vertex-weighted graph. Assume that \bar{G} is non-hyperelliptic and of genus 3. Then, for any divisor class δ on G , we have $r_{\bar{G}}^{\text{alg}}(\delta) \geq r_{\bar{G}}(\delta)$.*

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These results, combined with the above result of Caporaso–Len–Melo, show that the algebraic rank equals the combinatorial rank on all hyperelliptic vertex-weighted graphs (when $\text{char}(k) \neq 2$) and non-hyperelliptic vertex-weighted graphs of genus 3 (and certain graphs which are built from hyperelliptic vertex-weighted graphs and vertex-weighted graphs of genus at most 3; see Remark 5.3).

Caporaso [9, Conjecture 2.1] conjectured that the algebraic rank equals the combinatorial rank on any vertex-weighted graphs. It turns out that this is not the case in general; In [10], Caporaso, Len and Melo have found counterexamples, which we have learned while preparing this article. Since there are many graphs on which the algebraic rank equals the combinatorial rank (cf. Remark 5.3), it will be an interesting question to characterize such graphs.

To prove Theorem 1.1, we study the algebraic and combinatorial ranks of vertex-weighted graphs with a bridge. (For the definition of $\text{Bs}(\lfloor \underline{d}_i \rfloor^\bullet)$, see Section 2.2.)

Proposition 1.3. *Let $\bar{G} = (G, \omega)$ be a vertex-weighted graph having a bridge e with endpoints v_1, v_2 . Let G_1 and G_2 be the connected components of $G \setminus \{e\}$ such that $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, and set $\bar{G}_i = (G_i, \omega|_{V(G_i)})$ for $i = 1, 2$. Let $\underline{d} \in \text{Div}(G)$, and let $\underline{d}_i \in \text{Div}(G_i)$ be the restriction of \underline{d} to G_i . Then we have*

$$(1.1) \quad r_{\bar{G}}(\underline{d}) \leq \begin{cases} r_{\bar{G}_1}(\underline{d}_1) + r_{\bar{G}_2}(\underline{d}_2) + 1 & (\text{if } v_i \in \text{Bs}(\lfloor \underline{d}_i \rfloor^\bullet) \text{ for each } i = 1, 2), \\ r_{\bar{G}_1}(\underline{d}_1) + r_{\bar{G}_2}(\underline{d}_2) & (\text{otherwise}). \end{cases}$$

There is a formula corresponding to (1.1) (with the inequality replaced by the equality) for nodal curves (see Lemma 3.3). We prove Theorem 1.1 by induction on the number of bridges, using Proposition 1.3, Lemma 3.3 and [12, Corollary 1.7].

To prove Theorem 1.2, we show the following proposition. (See Section 2.4 for the notation.)

Proposition 1.4. *Let R be a complete discrete valuation ring with fractional field \mathbb{K} and residue field k . Let $\bar{G} = (G, \omega)$ be a non-hyperelliptic graph of genus 3. Let \mathcal{X} be a regular, generically smooth, semi-stable R -curve with reduction graph \bar{G} . Then the following condition (F) holds.*

$$(F) \quad \text{For any } \underline{d} \in \text{Div}(G), \text{ there exists a divisor } \tilde{D} \in \text{Div}(\mathcal{X}_{\mathbb{K}}) \text{ such that } \tilde{\rho}_*(\tilde{D}) = \underline{d} \text{ and } r_{\bar{G}}(\underline{d}) = r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D}),$$

where $\mathcal{X}_{\mathbb{K}}$ is the generic fiber of \mathcal{X} and $\tilde{\rho}_* : \text{Div}(\mathcal{X}_{\mathbb{K}}) \rightarrow \text{Div}(G)$ is the specialization map defined in (2.2).

We remark that a similar result for hyperelliptic graphs under a necessary assumption on their bridges is obtained in [12, Theorem 8.2]. (See Remark 5.2. See also Proposition 5.1 for a related result, which says that any non-hyperelliptic graph of genus 3 satisfies the condition (C) in [12].) The proof of Proposition 1.4 uses the specialization lemma of Amini–Caporaso [2, Theorem 4.10], which is based on Baker’s specialization lemma [4], and Raynaud’s theorem on the surjectivity of the specialization map between principal divisors (see [14], [4, Corollary A2] and Theorem A.1). Then we deduce Theorem 1.2 from Proposition 1.4 by the same argument as that in [12], which is due to Caporaso.

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2. COMBINATORIAL AND ALGEBRAIC RANKS OF DIVISORS ON GRAPHS

In this section, we recall definitions and properties of combinatorial and algebraic ranks of divisors on graphs, which will be used later.

2.1. Divisors on finite graphs. We briefly recall the theory of divisors on finite graphs. Our basic references are [5] and [6].

Throughout this paper, a *finite graph* means an unweighted, finite connected graph. We allow a finite graph to have loops and multiple edges. For a finite graph G , let $V(G)$ denote the set of vertices, and $E(G)$ the set of edges. The *genus* of G is defined as $g(G) = |E(G)| - |V(G)| + 1$. An edge $e \in E(G)$ is called a *bridge* if the deletion of e makes G disconnected.

Let $\text{Div}(G)$ be the free abelian group generated by $V(G)$. We call the elements of $\text{Div}(G)$ *divisors* on G . Any divisor $\underline{d} \in \text{Div}(G)$ is uniquely written as $\underline{d} = \sum_{v \in V(G)} n_v [v]$ for $n_v \in \mathbb{Z}$. The coefficient n_v at $[v]$ is denoted by $\underline{d}(v)$. A divisor \underline{d} is *effective*, written as $\underline{d} \geq 0$, if $\underline{d}(v) \geq 0$ for any $v \in V(G)$. The *degree* of a divisor \underline{d} is defined as $\deg(\underline{d}) = \sum_{v \in V(G)} \underline{d}(v)$.

A *rational function* on G is an integer-valued function on $V(G)$. We denote by $\text{Rat}(G)$ the set of rational functions on G . For $f \in \text{Rat}(G)$ and a vertex v of G , we set $\text{ord}_v(f) = \sum_{e=\overline{wv} \in E(G)} (f(w) - f(v))$, where the e 's run through all the edges of G with endpoint v . Then

$$\text{div}(f) := \sum_{v \in V(G)} \text{ord}_v(f) [v]$$

is a divisor on G . The set of *principal divisors* on G is defined as $\text{Prin}(G) = \{\text{div}(f) \mid f \in \text{Rat}(G)\}$. Then $\text{Prin}(G)$ is a subgroup of $\text{Div}(G)$, and we write $\text{Pic}(G) = \text{Div}(G) / \text{Prin}(G)$. For a divisor $\underline{d} \in \text{Div}(G)$, let $\text{cl}(\underline{d})$ denote its divisor class in $\text{Pic}(G)$.

Two divisors $\underline{d}, \underline{d}' \in \text{Div}(G)$ are said to be *linearly equivalent*, expressed as $\underline{d} \sim \underline{d}'$, if $\underline{d} - \underline{d}' \in \text{Prin}(G)$. For $\underline{d} \in \text{Div}(G)$, the complete linear system $|\underline{d}|$ is defined by

$$|\underline{d}| = \{\underline{d}' \in \text{Div}(G) \mid \underline{d}' \geq 0, \quad \underline{d}' \sim \underline{d}\}.$$

Definition 2.1 ((Combinatorial) rank of a divisor [5]). Let G be a finite graph. Let $\underline{d} \in \text{Div}(G)$. If $|\underline{d}| = \emptyset$, then we set $r_G(\underline{d}) := -1$. If $|\underline{d}| \neq \emptyset$, we set

$$r_G(\underline{d}) := \max \{s \in \mathbb{Z}_{\geq 0} \mid |\underline{d} - \underline{e}| \neq \emptyset \text{ for any effective divisor } \underline{e} \text{ with } \deg(\underline{e}) = s\}.$$

We note that $r_G(\underline{d})$ depends only on the divisor class of \underline{d} . For $\delta = \text{cl}(\underline{d}) \in \text{Pic}(G)$, we set $r_G(\delta) := r_G(\underline{d})$.

A vertex $v \in V(G)$ is called a *base-point* of the complete linear system $|\underline{d}|$ if $r_G(\underline{d} - [v]) = r_G(\underline{d})$. The set of base-points of $|\underline{d}|$ is denoted by $\text{Bs}(|\underline{d}|)$. If $|\underline{d}| = \emptyset$, then any vertex of G is a base-point of $|\underline{d}|$ by definition.

In the rest of this subsection, we assume that G is loopless. For any subset $A \subseteq V(G)$ and $v \in V(G)$, the *out-degree* of v from A , denoted by $\text{outdeg}_A(v)$, is the number of edges of G having v as one endpoint and whose other endpoint lies in $V(G) \setminus A$. For $\underline{d} \in \text{Div}(G)$, a vertex $v \in A$ is *saturated* for \underline{d} with respect to A if $\underline{d}(v) \geq \text{outdeg}_A(v)$, and *non-saturated* otherwise.

Definition 2.2 (v_0 -reduced divisor [5]). Fix a base vertex $v_0 \in V(G)$. A divisor $\underline{d} \in \text{Div}(G)$ is called a *v_0 -reduced divisor* if $\underline{d}(v) \geq 0$ for any $v \in V(G) \setminus \{v_0\}$, and every non-empty subset A of $V(G) \setminus \{v_0\}$ contains a non-saturated vertex $v \in A$ for \underline{d} with respect to A .

We recall from [5] key properties of v_0 -reduced divisors, which will be used later.

Proposition 2.3 ([5, Proposition 3.1 and its proof]). *Fix a base vertex $v_0 \in V(G)$. Then for any $\underline{d} \in \text{Div}(G)$, there exists a unique v_0 -reduced divisor $\underline{d}' \in \text{Div}(G)$ that is linearly equivalent to \underline{d} . Further, $r_G(\underline{d}) \geq 0$ if and only if \underline{d}' is effective.*

The canonical divisor K_G on G is defined by $K_G = \sum_{v \in V(G)} (\text{val}(v) - 2)[v] \in \text{Div}(G)$, where $\text{val}(v)$ denotes the number of edges with endpoint v . We remark that, with the above definition of rank, the notion of v_0 -reduced divisors and the canonical divisor on G , Baker and Norine [5, Theorem 1.12] established the Riemann–Roch theorem on a loopless finite graph.

Finally, we recall the definition of hyperelliptic graphs.

Definition 2.4 (Hyperelliptic graph [6]). A loopless finite graph G of $g(G) \geq 2$ is said to be *hyperelliptic* if there exists a divisor $\underline{d} \in \text{Div}(G)$ such that $\deg(\underline{d}) = 2$ and $r_{\bar{G}}(\underline{d}) = 1$.

2.2. Rank of divisors on vertex-weighted graphs. We briefly recall the theory of divisors on vertex-weighted graphs. Our basic references are [2] and [9].

A *vertex-weighted* graph $\bar{G} = (G, \omega)$ is the pair of a finite graph G and a function (called a vertex-weight function) $\omega : V(G) \rightarrow \mathbb{Z}_{\geq 0}$. The genus of \bar{G} is defined as $g(\bar{G}) = g(G) + \sum_{v \in V(G)} \omega(v)$.

For a vertex-weighted graph $\bar{G} = (G, \omega)$, we make a loopless finite graph \bar{G}^\bullet as follows: We add $\omega(v)$ loops to G at v for every vertex $v \in V(G)$; Then we insert a vertex in every loop edge. The graph \bar{G}^\bullet is called *the virtual loopless finite graph of \bar{G}* .

We have natural embeddings of the vertices $V(G) \subseteq V(\bar{G}^\bullet)$, and of the divisor groups $\text{Div}(G) \subseteq \text{Div}(\bar{G}^\bullet)$. For $\underline{d} \in \text{Div}(G)$, the rank $r_{\bar{G}}(\underline{d})$ of \underline{d} is defined by

$$r_{\bar{G}}(\underline{d}) := r_{\bar{G}^\bullet}(\underline{d}),$$

where the right-hand side is defined in Definition 2.1. Since $\text{Prin}(G) \subseteq \text{Prin}(\bar{G}^\bullet)$, $r_{\bar{G}}(\underline{d})$ depends only on the divisor class of \underline{d} . For $\delta = \text{cl}(\underline{d}) \in \text{Pic}(G)$, we set $r_{\bar{G}}(\delta) := r_{\bar{G}}(\underline{d})$.

For $\underline{d} \in \text{Div}(G)$, we write $|\underline{d}|^\bullet$ for the complete linear system $|\underline{d}|$ on \bar{G}^\bullet . Namely, we have

$$|\underline{d}|^\bullet := \{\underline{d}' \in \text{Div}(\bar{G}^\bullet) \mid \underline{d}' \geq 0, \underline{d}' \text{ is linearly equivalent to } \underline{d} \text{ in } \bar{G}^\bullet\}.$$

Here we use the notation “ \bullet ” to emphasize that we are considering divisors on \bar{G}^\bullet .

Let $K_{\bar{G}^\bullet}$ be the canonical divisor of \bar{G}^\bullet . Then the support of $K_{\bar{G}^\bullet}$ lies in $V(G)$. We regard $K_{\bar{G}^\bullet}$ as an element of $\text{Div}(G)$, and we define the *canonical divisor* $K_{\bar{G}}$ of \bar{G} by $K_{\bar{G}} := K_{\bar{G}^\bullet} \in \text{Div}(G)$. We remark that, if G is loopless, then $K_{\bar{G}} = K_G + \sum_{v \in V(G)} 2\omega(v)[v]$.

A vertex-weighted graph \bar{G} of $g(\bar{G}) \geq 2$ is said to be *hyperelliptic* if its virtual loopless finite graph \bar{G}^\bullet is hyperelliptic.

2.3. Algebraic rank. Following [9], we recall the notion of the algebraic rank of a divisor class δ on a vertex-weighted graph.

Let k be a fixed algebraically closed field. By a *nodal curve*, we mean a connected, reduced, projective, one dimensional scheme over k with at most ordinary double points as singularities.

For a nodal curve X , the group of Cartier divisors is denoted by $\text{Div}(X)$. We set $\text{Pic}(X) = \text{Div}(X)/\text{Prin}(X)$, where $\text{Prin}(X)$ denotes the group of principal divisors. For $L \in \text{Pic}(X)$, we write $r_X(L) = \dim_k H^0(X, L) - 1$.

Given a nodal curve X , the (vertex-weighted) *dual graph* $\bar{G} = (G, \omega)$ associated to X is defined as follows. Let C_1, \dots, C_r be the irreducible components of X . Then G has vertices v_1, \dots, v_r which correspond to C_1, \dots, C_r , respectively. Two vertices v_i, v_j

$(i \neq j)$ of G are connected by a_{ij} edges if $\#C_i \cap C_j = a_{ij}$. A vertex v_i has b_i loops if $\#\text{Sing}(C_i) = b_i$. The vertex-weighted function ω is given by assigning to v_i the geometric genus of X_i .

Let $\bar{G} = (G, \omega)$ be a vertex-weighted graph. Let $M^{\text{alg}}(\bar{G})$ be a family of nodal curves representing all the isomorphism classes of nodal curves with dual graph \bar{G} . For $X \in M^{\text{alg}}(\bar{G})$, we write $X = \cup_{v \in V(G)} C_v$, where C_v is the irreducible curve corresponding to $v \in V(G)$. We have a natural map

$$(2.1) \quad \rho_* : \text{Div}(X) \rightarrow \text{Div}(G), \quad D \mapsto \sum_{v \in V(G)} (\deg(D|_{C_v})) [v].$$

In other words, for a Cartier divisor D on X , $\rho_*(D) \in \text{Div}(G)$ gives the multidegree of D . Since linear equivalent divisors on X have the same multidegree, ρ_* descends to $\text{Pic}(X) \rightarrow \text{Div}(G)$. Then we have a stratification of $\text{Pic}(X)$:

$$\text{Pic}(X) = \bigsqcup_{\underline{d} \in \text{Div}(G)} \text{Pic}^{\underline{d}}(X),$$

where $\text{Pic}^{\underline{d}}(X) = \{L \in \text{Pic}(X) \mid \deg(L|_{C_v}) = d_v \text{ for any } v \in V(G)\}$ for $\underline{d} = (d_v)_{v \in V(G)} \in \text{Div}(G)$.

Definition 2.5 (Algebraic rank [9]). Let $\bar{G} = (G, \omega)$ be a vertex-weighted graph, and $\delta \in \text{Pic}(G)$ a divisor class on G . We set

$$r_G^{\text{alg}}(\delta) = \max_{X \in M^{\text{alg}}(\bar{G})} \left\{ \min_{\underline{d} \in \delta} \left\{ \max_{L \in \text{Pic}^{\underline{d}}(X)} \{r_X(L)\} \right\} \right\},$$

and call $r_G^{\text{alg}}(\delta)$ the *algebraic rank* of combinatorial type δ .

2.4. The specialization lemma for vertex-weighted graphs. We recall the specialization lemma for vertex-weighted graphs due to Amini–Caporaso [2], which generalizes Baker’s specialization lemma for loopless finite graphs [4]. Our basic references are [2] and [4].

Let k be a fixed algebraically closed field. Let R be a complete discrete valuation ring with residue field k . Let \mathbb{K} denote the fractional field of R .

By an *R-curve*, we mean an integral scheme of dimension 2 that is proper and flat over $\text{Spec}(R)$. For an *R-curve* \mathcal{X} , we denote by $\mathcal{X}_{\mathbb{K}}$ the generic fiber of \mathcal{X} , and by X the special fiber of \mathcal{X} . We say that \mathcal{X} is a *semi-stable R-curve* if X is a nodal curve. The vertex-weighted dual graph $\bar{G} = (G, \omega)$ of X is then called the *reduction graph* of \mathcal{X} .

Let \mathcal{X} be a regular, generically smooth, semi-stable *R-curve*. Since $\mathcal{X}_{\mathbb{K}}$ is smooth (resp. \mathcal{X} is regular), the group of Cartier divisors on $\mathcal{X}_{\mathbb{K}}$ (resp. \mathcal{X}) is the same as the group of Weil divisors. The Zariski closure of an effective divisor on $\mathcal{X}_{\mathbb{K}}$ in \mathcal{X} is a Cartier divisor. Extending by linearity, one can associate to any divisor on $\mathcal{X}_{\mathbb{K}}$ a Cartier divisor on \mathcal{X} , which is also called the Zariski closure of the divisor.

Let \tilde{D} be a divisor on $\mathcal{X}_{\mathbb{K}}$ and $\tilde{\mathcal{D}}$ the Zariski closure of \tilde{D} . Let $\mathcal{O}_{\mathcal{X}}(\tilde{\mathcal{D}})$ be the invertible sheaf on \mathcal{X} associated to $\tilde{\mathcal{D}}$. We define the *specialization map* $\tilde{\rho}_* : \text{Div}(\mathcal{X}_{\mathbb{K}}) \rightarrow \text{Div}(G)$ by

$$(2.2) \quad \tilde{\rho}_*(\tilde{D}) := \sum_{v \in V(G)} \deg(\mathcal{O}_{\mathcal{X}}(\tilde{\mathcal{D}})|_{C_v}) [v] \in \text{Div}(G)$$

(see [4, §2.1]). The map $\tilde{\rho}_*$ is compatible with the map ρ_* in (2.1): Namely, let $D \in \text{Div}(X)$ be a Cartier divisor on the special fiber such that the associated invertible sheaf $\mathcal{O}_X(D)$

is isomorphic to $\mathcal{O}_{\mathcal{X}}(\tilde{\mathcal{D}})|_X$; Then, by definition, we have

$$(2.3) \quad \rho_*(D) = \tilde{\rho}_*(\tilde{D}).$$

Remark 2.6. In [12], $\tilde{\rho}_*$ is denoted by ρ_* . Here we use the notation $\tilde{\rho}_*$, for we have already use the notation ρ_* in (2.1).

Let $\text{Div}(\mathcal{X}(\mathbb{K}))$ be the subgroup of $\text{Div}(\mathcal{X}_{\mathbb{K}})$ generated by \mathbb{K} -valued points of \mathcal{X} . Then

$$(2.4) \quad \tilde{\rho}_*|_{\text{Div}(\mathcal{X}_{\mathbb{K}})} : \text{Div}(\mathcal{X}(\mathbb{K})) \rightarrow \text{Div}(G)$$

is surjective (see [4, Remark 2.3] and [13, Proposition 10.1.40(b)]).

Theorem 2.7 (Amini–Caporaso’s specialization lemma [2, Theorem 4.10]). *Let \mathcal{X} be a regular, generically smooth, semi-stable R -curve with reduction graph $\bar{G} = (G, \omega)$. Then, for any $\tilde{D} \in \text{Div}(\mathcal{X}_{\mathbb{K}})$, one has $r_{\bar{G}}(\tilde{\rho}_*(\tilde{D})) \geq r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D})$.*

Theorem 2.7 is a generalization of Baker’s specialization lemma for loopless finite graphs in [4]. Although Amini and Caporaso consider a smooth quasi-projective curve B over k (in place of $\text{Spec}(R)$), i.e., they consider a morphism $\phi : \mathcal{X} \rightarrow B$, we remark that their arguments also work over $\text{Spec}(R)$. (By the surjectivity of the map (2.4), the argument over R works as the same as the argument for $\phi : \mathcal{X} \rightarrow B$ which admits a section passing through any given component of the special fiber.)

3. REDUCED DIVISORS AND DECOMPOSITION OF GRAPHS

In this section, we prove Proposition 1.3. We first show some properties of divisors on a graph with a bridge.

Lemma 3.1. *Let G be a loopless finite graph with a bridge e having endpoints v_1, v_2 . Let G_1 and G_2 be the connected components of $G \setminus \{e\}$ such that $v_1 \in V(G_1), v_2 \in V(G_2)$. For $i = 1, 2$, let $j_i : V(G_i) \hookrightarrow V(G)$ be the natural embedding and $j_{i*} : \text{Div}(G_i) \hookrightarrow \text{Div}(G)$ the induced map.*

- (1) *For $i = 1, 2$, we have $j_{i*}(\text{Prin}(G_i)) \subseteq \text{Prin}(G)$.*
- (2) *For $i = 1, 2$, let \underline{d}_i be a v_i -reduced divisor on G_i . Then*

$$(3.1) \quad j_{1*}(\underline{d}_1 - \underline{d}_1(v_1)[v_1]) + j_{2*}(\underline{d}_2 - \underline{d}_2(v_2)[v_2]) + (\underline{d}_1(v_1) + \underline{d}_2(v_2))[v_1]$$

is a v_1 -reduced divisor on G .

Proof. (1) We may assume that $i = 1$. Let f_1 be a rational function on G_1 . We extend f_1 to a rational function \tilde{f}_1 on G by setting $\tilde{f}_1(w) = f_1(v_1)$ for any $w \in V(G_2)$. Then we have $\text{div}(\tilde{f}) = j_{1*}(\text{div}(f_1))$. Thus $j_{1*}(\text{div}(f_1)) \in \text{Prin}(G)$, which gives the assertion.

(2) We put

$$\underline{d} := j_{1*}(\underline{d}_1 - \underline{d}_1(v_1)[v_1]) + j_{2*}(\underline{d}_2 - \underline{d}_2(v_2)[v_2]).$$

It suffices to show that \underline{d} is a v_1 -reduced divisor on G . Let $A \subseteq V(G) \setminus \{v_1\}$ be any non-empty subset, and we are going to show that there exists a non-saturated vertex $v \in A$ for \underline{d} with respect to A .

If $v_2 \in A$, then it follows from $v_1 \notin A$ that $\text{outdeg}_A(v_2) \geq 1$ (from the contribution of the bridge e). Since $\underline{d}(v_2) = 0$, we see that $v_2 \in V(G) \setminus \{v_1\}$ is a non-saturated vertex for \underline{d} with respect to A . Thus we may and do assume that $v_2 \notin A$, and hence $A \subseteq V(G) \setminus \{v_1, v_2\}$.

We set $A_1 := A \cap V(G_1)$ and $A_2 := A \cap V(G_2)$. Then $A_1 \subseteq V(G) \setminus \{v_1\}$ and $A_2 \subseteq V(G) \setminus \{v_2\}$. Since $A \neq \emptyset$, we have $A_1 \neq \emptyset$ or $A_2 \neq \emptyset$. Without loss of generality, we

assume that $A_1 \neq \emptyset$. Since \underline{d}_1 is a v_1 -reduced divisor on G_1 , there exists a non-saturated vertex $v \in A_1$ for \underline{d}_1 with respect to A_1 , i.e., $\underline{d}_1(v) < \text{outdeg}_{A_1}(v)$. Since $\underline{d}_1(v) = \underline{d}(v)$ and $\text{outdeg}_{A_1}(v) = \text{outdeg}_A(v)$, we have $\underline{d}(v) < \text{outdeg}_A(v)$. Thus $v \in A$ is a non-saturated vertex for \underline{d} with respect to A , which shows the lemma. \square

The next lemma will be used in Section 5.

Lemma 3.2. *Let $\bar{G} = (G, \omega)$ be a vertex-weighted graph. Let $\underline{d} \in \text{Div}(G)$. If $r_{\bar{G}}(\underline{d}) \geq 0$, then there exists an effective divisor $\underline{e} \in \text{Div}(G)$ that is linearly equivalent to \underline{d} in G .*

Proof. Let \bar{G}^\bullet be the virtual loopless finite graph of \bar{G} . Via the natural embedding of the sets of vertices, we regard $V(G) \subseteq V(\bar{G}^\bullet)$. The condition $r_{\bar{G}}(\underline{d}) := r_{\bar{G}^\bullet}(\underline{d}) \geq 0$ means that there exists a rational function $\bar{f} \in \text{Rat}(\bar{G}^\bullet)$ such that $\underline{d}' := \underline{d} + \text{div}(\bar{f})$ is an effective divisor on \bar{G}^\bullet .

Let $w \in V(\bar{G}^\bullet) \setminus V(G)$. This means that w is a vertex inserted in a loop edge. Thus there exist exactly two edges e_1, e_2 of \bar{G}^\bullet with endpoint w , and the other endpoint of e_1 and that of e_2 are the same, which we denote by w' . Since $\underline{d}(w) = 0$ and $\underline{d}'(w) \geq 0$, we see that $\bar{f}(w') \geq \bar{f}(w)$.

We set $f := \bar{f}|_{V(G)} \in \text{Rat}(G)$. Since $\underline{d} \in \text{Div}(G)$ and $\bar{f}(w') \geq \bar{f}(w)$ for every $w \in V(\bar{G}^\bullet) \setminus V(G)$, we see that $\underline{e} := \underline{d} + \text{div}(f)$ is an effective divisor on G . This shows the lemma. \square

We begin the proof of Proposition 1.3.

Proof of Proposition 1.3. Let $\bar{G}^\bullet, \bar{G}_1^\bullet$ and \bar{G}_2^\bullet be the virtual loopless finite graphs of \bar{G}, \bar{G}_1 and \bar{G}_2 , respectively. Note that \bar{G}^\bullet is the graph obtained by connecting \bar{G}_1^\bullet and \bar{G}_2^\bullet with the edge e . For $i = 1, 2$, let $j_{i*}^\bullet : \text{Div}(\bar{G}_i^\bullet) \hookrightarrow \text{Div}(\bar{G}^\bullet)$ be the induced embedding of divisors.

Via the natural embedding of the sets of vertices, we regard $V(G_i) \subseteq V(G) \subseteq V(\bar{G}^\bullet)$ and $V(G_i) \subseteq V(\bar{G}_i^\bullet) \subseteq V(\bar{G}^\bullet)$ for $i = 1, 2$. Thus, in the following argument, we will often identify the vertex $v_i \in V(G_i)$ with the corresponding vertices in G, \bar{G}_i^\bullet and \bar{G}^\bullet .

For $i = 1, 2$, we set $r_i = r_{\bar{G}_i^\bullet}(\underline{d}_i)$. By the definition of the rank, there exists an effective divisor $\underline{e}_i \in \text{Div}(\bar{G}_i^\bullet)$ with $\deg(\underline{e}_i) = r_i + 1$ such that $r_{\bar{G}_i^\bullet}(\underline{d}_i - \underline{e}_i) = -1$. We set $\underline{f}_i = \underline{d}_i - \underline{e}_i$, and let $\underline{f}_i' \in \text{Div}(\bar{G}_i^\bullet)$ be the v_i -reduced divisor that is linearly equivalent to \underline{f}_i on \bar{G}_i^\bullet . Since $r_{\bar{G}_i^\bullet}(\underline{d}_i - \underline{e}_i) = -1$, we have $\underline{f}_i'(v_i) < 0$ by Proposition 2.3.

We claim that $r_{\bar{G}^\bullet}(\underline{d} - j_{1*}^\bullet(\underline{e}_1) - j_{2*}^\bullet(\underline{e}_2)) = -1$. Indeed, we see from Lemma 3.1(1) that, as divisors on \bar{G}^\bullet ,

$$\begin{aligned} & \underline{d} - j_{1*}^\bullet(\underline{e}_1) - j_{2*}^\bullet(\underline{e}_2) \\ &= j_{1*}^\bullet(\underline{d}_1 - \underline{e}_1) + j_{2*}^\bullet(\underline{d}_2 - \underline{e}_2) \\ &\sim j_{1*}^\bullet(\underline{f}_1') + j_{2*}^\bullet(\underline{f}_2') \\ &= j_{1*}^\bullet(\underline{f}_1' - \underline{f}_1'(v_1)[v_1]) + j_{2*}^\bullet(\underline{f}_2' - \underline{f}_2'(v_2)[v_2]) + (\underline{f}_1'(v_1) + \underline{f}_2'(v_2))[v_1]. \end{aligned}$$

We denote by \underline{g} the divisor in the last line in the above. Then, by Lemma 3.1(2), \underline{g} is a v_1 -reduced divisor on \bar{G}^\bullet . Since $\underline{g}(v_1) = \underline{f}_1'(v_1) + \underline{f}_2'(v_2) < 0$, we have $r_{\bar{G}^\bullet}(\underline{d} - j_{1*}^\bullet(\underline{e}_1) - j_{2*}^\bullet(\underline{e}_2)) = -1$ by Proposition 2.3.

It follows that

$$\begin{aligned} r_{\bar{G}}(\underline{d}) &= r_{\bar{G}^\bullet}(\underline{d}) \leq \deg(j_{1*}^\bullet(\underline{e}_1) + j_{2*}^\bullet(\underline{e}_2)) - 1 \\ &= r_1 + r_2 + 1 = r_{\bar{G}_1}(d_1) + r_{\bar{G}_2}(d_2) + 1. \end{aligned}$$

This shows the inequality in (1.1) in the case of $v_i \in \text{Bs}(|\underline{d}_i|^\bullet)$ for each $i = 1, 2$.

Suppose now that $v_1 \notin \text{Bs}(|\underline{d}_1|^\bullet)$ or $v_2 \notin \text{Bs}(|\underline{d}_2|^\bullet)$. We need to show that

$$r_{\bar{G}}(\underline{d}) \leq r_{\bar{G}_1}(\underline{d}_1) + r_{\bar{G}_2}(\underline{d}_2).$$

Without loss of generality, we may assume that $v_2 \notin \text{Bs}(|\underline{d}_2|^\bullet)$. This means that $r_{\bar{G}_2}(\underline{d}_2 - [v_2]) = r_2 - 1$. Note that $r_2 \geq 0$.

By the definition of the rank, there exists an effective divisor $\tilde{e}_2 \in \text{Div}(\bar{G}_2^\bullet)$ with $\deg(\tilde{e}_2) = r_2$ such that $r_{\bar{G}_2^\bullet}(\underline{d}_2 - [v_2] - \tilde{e}_2) = -1$. We set $\underline{h}_2 = \underline{d}_2 - \tilde{e}_2$, and let $\underline{h}_2' \in \text{Div}(\bar{G}_2^\bullet)$ be the v_2 -reduced divisor that is linearly equivalent to \underline{h}_2 on \bar{G}_2^\bullet . Since $r_{\bar{G}_2}(\underline{d}_2) = r_2$, we have $r_{\bar{G}_2^\bullet}(\underline{d}_2 - \tilde{e}_2) \geq 0$, hence \underline{h}_2' is an effective divisor on \bar{G}_2^\bullet by Proposition 2.3. Since \underline{h}_2' is v_2 -reduced and since $r_{\bar{G}_2^\bullet}(\underline{h}_2' - [v_2]) = r_{\bar{G}_2^\bullet}(\underline{d}_2 - [v_2] - \tilde{e}_2) = -1$, we see that $\underline{h}_2'(v_2) = 0$.

It follows from Lemma 3.1(1) that

$$\begin{aligned} \underline{d} - j_{1*}^\bullet(\underline{e}_1) - j_{2*}^\bullet(\tilde{e}_2) &= j_{1*}^\bullet(\underline{d}_1 - \underline{e}_1) + j_{2*}^\bullet(\underline{d}_2 - \tilde{e}_2) \\ &\sim j_{1*}^\bullet(\underline{f}_1') + j_{2*}^\bullet(\underline{h}_2'). \end{aligned}$$

Since $\underline{h}_2'(v_2) = 0$, we see that $j_{1*}^\bullet(\underline{f}_1') + j_{2*}^\bullet(\underline{h}_2')$ is a v_1 -reduced divisor on \bar{G}^\bullet by Lemma 3.1(2). Since $\underline{f}_1'(v_1) + \underline{h}_2'(v_2) = \underline{f}_1'(v_1) < 0$, Proposition 2.3 tells us that $r_{\bar{G}^\bullet}(\underline{d} - j_{1*}^\bullet(\underline{e}_1) - j_{2*}^\bullet(\tilde{e}_2)) = -1$. Then

$$\begin{aligned} r_{\bar{G}}(\underline{d}) &= r_{\bar{G}^\bullet}(\underline{d}) \leq \deg(j_{1*}^\bullet(\underline{e}_1) + j_{2*}^\bullet(\tilde{e}_2)) - 1 \\ &= r_1 + r_2 = r_{\bar{G}_1}(\underline{d}_1) + r_{\bar{G}_2}(\underline{d}_2). \end{aligned}$$

Thus we obtain the inequality in the remaining case. \square

There exists a formula corresponding to Proposition 1.3 (with the inequality replaced by the equality) for nodal curves.

Lemma 3.3. *Let X be a nodal curve. We assume that X has a decomposition as $X = X_1 \cup X_2$ into two nodal curves so that X_1 and X_2 meet at exactly one point p . Let D be a Cartier divisor on X , and we set $D_i = D|_{X_i} \in \text{Div}(X_i)$ for $i = 1, 2$. Then*

$$(3.2) \quad r_X(D) = \begin{cases} r_{X_1}(D_1) + r_{X_2}(D_2) + 1 & (\text{if } p \in \text{Bs}(|D_i|) \text{ for each } i = 1, 2), \\ r_{X_1}(D_1) + r_{X_2}(D_2) & (\text{otherwise}). \end{cases}$$

Proof. This is a well-known fact, so we omit a proof. See also [9, Remark 1.5]. \square

The following simple remark will be used in the next section.

Remark 3.4. Let X, X_1, X_2, p be as in Lemma 3.3. For $i = 1, 2$, let D_i be a Cartier divisor on X_i . Then there exists a Cartier divisor D on X such that $D|_{X_i}$ is linearly equivalent to D_i . Indeed, let $p_i : X \rightarrow X_i$ be the morphism given by the identity on X_i and the constant map to p on the other component. Let $\mathcal{O}_{X_i}(D_i)$ be the invertible sheaf on X_i associated to D_i . Then it suffices to take $D \in \text{Div}(X)$ such that the associated invertible sheaf $\mathcal{O}_X(D)$ is isomorphic to $p_1^*(\mathcal{O}_{X_1}(D_1)) \otimes p_2^*(\mathcal{O}_{X_2}(D_2))$.

4. GRAPHS WITH A BRIDGE AND HYPERELLIPTIC GRAPHS

In this section, we prove Theorem 1.1. We begin by showing the following lemma.

Lemma 4.1. *Let $\bar{G} = (G, \omega)$ be a vertex-weighted graph with a bridge e with endpoints v_1, v_2 . Let G_1 and G_2 be the connected components of $G \setminus \{e\}$ such that $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, and we set $\bar{G}_i = (G_i, \omega|_{V(G_i)})$ for $i = 1, 2$. Let $\underline{d} \in \text{Div}(G)$, and let $\underline{d}_i \in \text{Div}(G_i)$ be the restriction of \underline{d} to G_i . Let X be a nodal curve over k with dual graph \bar{G} , and we write X_i for the union of irreducible components of X corresponding to \bar{G}_i . Let $\rho_* : \text{Div}(X) \rightarrow \text{Div}(G)$ and $\rho_{i*} : \text{Div}(X_i) \rightarrow \text{Div}(G_i)$ be the maps defined in (2.1). For $i = 1, 2$, we assume that, for any divisor $\underline{e}_i \in \text{Div}(G_i)$, there exists a Cartier divisor E_i on X_i satisfying $\rho_{i*}(E_i) = \underline{e}_i$ and $r_{X_i}(E_i) \geq r_{\bar{G}_i}(\underline{e}_i)$. Then there exists a Cartier divisor D on X satisfying $\rho_*(D) = \underline{d}$ and*

$$r_X(D) \geq \begin{cases} r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 2 & (\text{if } v_i \notin \text{Bs}(|\underline{d}_i|^\bullet) \text{ for each } i = 1, 2), \\ r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 1 & (\text{otherwise}). \end{cases}$$

Proof. **Case 1.** Suppose that $v_i \notin \text{Bs}(|\underline{d}_i|^\bullet)$ for each $i = 1, 2$. This means that $|\underline{d}_i|^\bullet \neq \emptyset$ and

$$(4.1) \quad r_{\bar{G}_i}(\underline{d}_i - [v_i]) = r_{\bar{G}_i}(\underline{d}_i) - 1.$$

We take a Cartier divisor D_i on X_i satisfying $\rho_{i*}(D_i) = \underline{d}_i$ and $r_{X_i}(D_i) \geq r_{\bar{G}_i}(\underline{d}_i)$. By Remark 3.4, there exists a Cartier divisor D on X such that $D|_{X_i}$ is linearly equivalent to D_i on X_i . Then we have

$$\begin{aligned} \rho_*(D) &= \rho_{1*}(D|_{X_1}) + \rho_{2*}(D|_{X_2}) \\ &= \rho_{1*}(D_1) + \rho_{2*}(D_2) = \underline{d}_1 + \underline{d}_2 = \underline{d}. \end{aligned}$$

Further, we have

$$\begin{aligned} r_X(D) &\geq r_{X_1}(D|_{X_1}) + r_{X_2}(D|_{X_2}) && (\text{from Lemma 3.3}) \\ &= r_{X_1}(D_1) + r_{X_2}(D_2) && (\text{since } D|_{X_i} \sim D_i \text{ for each } i = 1, 2) \\ &\geq r_{\bar{G}_1}(\underline{d}_1) + r_{\bar{G}_2}(\underline{d}_2) && (\text{from the assumptions on } D_i) \\ &= r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 2 && (\text{from (4.1)}). \end{aligned}$$

This gives the desired properties in this case.

Case 2. Suppose that $v_1 \in \text{Bs}(|\underline{d}_1|^\bullet)$ or $v_2 \in \text{Bs}(|\underline{d}_2|^\bullet)$. For $i = 1, 2$, we take a Cartier divisor D'_i on X_i satisfying $\rho_{i*}(D'_i) = \underline{d}_i - [v_i]$ and $r_{X_i}(D'_i) \geq r_{\bar{G}_i}(\underline{d}_i - [v_i])$.

We remark that $X = X_1 \cup X_2$ and that $X_1 \cap X_2$ consists of the node of X corresponding to the edge e . Let p denote this node. Since p is a smooth point on X_i , the Weil divisor $[p]$ is regarded as a Cartier divisor on X_i . We set

$$D_i = D'_i + [p] \in \text{Div}(X_i).$$

By Remark 3.4, there exists a Cartier divisor D on X such that $D|_{X_i}$ is linearly equivalent to D_i on X_i . Then we have $\rho_*(D) = \underline{d}$ as in Case 1.

For $i = 1, 2$, we set

$$\varepsilon_i = \begin{cases} 1 & (\text{if } p \in \text{Bs}(D_i)), \\ 0 & (\text{if } p \notin \text{Bs}(D_i)), \end{cases}$$

so that $r_{X_i}(D'_i) = r_X(D_i) - (1 - \varepsilon_i)$. Then it follows from Lemma 3.3 that

$$\begin{aligned}
r_X(D) &= r_{X_1}(D|_{X_1}) + r_{X_2}(D|_{X_2}) + \varepsilon_1\varepsilon_2 \\
&= r_{X_1}(D_1) + r_{X_2}(D_2) + \varepsilon_1\varepsilon_2 \\
&= r_{X_1}(D'_1) + r_{X_2}(D'_2) + (1 - \varepsilon_1) + (1 - \varepsilon_2) + \varepsilon_1\varepsilon_2 \\
&\geq r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 1 + (1 - \varepsilon_1)(1 - \varepsilon_2) \\
&\geq r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 1.
\end{aligned}$$

This gives the desired properties in the remaining case, thus completing the proof. \square

Next, we reinterpret Proposition 1.3.

Lemma 4.2. *In the setting of Proposition 1.3, we have*

(4.2)

$$r_{\bar{G}}(\underline{d}) \leq \begin{cases} r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 2 & (\text{if } v_i \notin \text{Bs}(|\underline{d}_i|^\bullet) \text{ for each } i = 1, 2), \\ r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 1 & (\text{otherwise}). \end{cases}$$

Proof. First we consider the case where $v_i \notin \text{Bs}(|\underline{d}_i|^\bullet)$ for each $i = 1, 2$. Then, since $r_{\bar{G}_i}(\underline{d}_i - [v_i]) = r_{\bar{G}_i}(\underline{d}_i) - 1$, it follows from Proposition 1.3 that

$$r_{\bar{G}}(\underline{d}) \leq r_{\bar{G}_1}(\underline{d}_1) + r_{\bar{G}_2}(\underline{d}_2) = r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 2.$$

This shows the inequality (4.2) in this case.

Next consider the case where $v_1 \in \text{Bs}(|\underline{d}_1|^\bullet)$ and $v_2 \notin \text{Bs}(|\underline{d}_2|^\bullet)$. (The case of $v_1 \notin \text{Bs}(|\underline{d}_1|^\bullet)$ and $v_2 \in \text{Bs}(|\underline{d}_2|^\bullet)$ is shown in the same way.) Then we have $r_{\bar{G}_1}(\underline{d}_1 - [v_1]) = r_{\bar{G}_1}(\underline{d}_1)$, and $r_{\bar{G}_2}(\underline{d}_2 - [v_2]) = r_{\bar{G}_2}(\underline{d}_2) - 1$. It follows from Proposition 1.3 that

$$r_{\bar{G}}(\underline{d}) \leq r_{\bar{G}_1}(\underline{d}_1) + r_{\bar{G}_2}(\underline{d}_2) = r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 1,$$

which shows (4.2) in this case.

Finally consider the case where $v_i \in \text{Bs}(|\underline{d}_i|^\bullet)$ for each $i = 1, 2$. This means that $r_{\bar{G}_i}(\underline{d}_i - [v_i]) = r_{\bar{G}_i}(\underline{d}_i)$, and Proposition 1.3 gives

$$r_{\bar{G}}(\underline{d}) \leq r_{\bar{G}_1}(\underline{d}_1) + r_{\bar{G}_2}(\underline{d}_2) + 1 = r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 1.$$

Thus we obtain (4.2). \square

To prove Theorem 1.1, we consider the following condition for a vertex-weighted graph \bar{G} .

(FS) There exists a nodal curve X with dual graph \bar{G} such that, for any $\underline{d} \in \text{Div}(G)$, there exists a Cartier divisor D on X such that $\rho_*(D) = \underline{d}$ and $r_X(D) \geq r_{\bar{G}}(\underline{d})$, where $\rho_* : \text{Div}(X) \rightarrow \text{Div}(G)$ is the map defined in (2.1).

We note that if a vertex-weighted graph \bar{G} satisfies the condition (FS), then we have $r_{\bar{G}}^{\text{alg}}(\delta) \geq r_{\bar{G}}(\delta)$ for any divisor class $\delta \in \text{Pic}(G)$. Indeed, let δ be any divisor class of G . We take the nodal curve X in the condition (FS). Then for any representative $\underline{d} \in \text{Div}(G)$ of δ , we take a Cartier divisor D on X as in (FS). With X as above, we obtain $\min_{\underline{d} \in \delta} \left\{ \max_{L \in \text{Pic}^{\underline{d}}(X)} \{r_X(L)\} \right\} \geq r_{\bar{G}}(\delta)$. Thus we get $r_{\bar{G}}^{\text{alg}}(\delta) \geq r_{\bar{G}}(\delta)$.

Proof of Theorem 1.1. Let \bar{G} be a hyperelliptic vertex-weighted graph. We will show that \bar{G} satisfies the condition (FS) by the induction on the number of bridges. As is explained as above, we will then have the desired inequality $r_{\bar{G}}^{\text{alg}}(\delta) \geq r_{\bar{G}}(\delta)$ for any divisor class δ on G .

If G has no bridges, then [12, Proposition 1.5 and its proof and Theorem 8.2] tells us that \bar{G} satisfies the condition (FS) (More generally, if there are at most $(2\omega(v) + 2)$ “positive-type” bridges emanating from each vertex $v \in V(G)$, then \bar{G} satisfies the condition (FS): See [12].) Also, a vertex-weighted graph of genus at most 1 satisfies the condition (FS) (cf. [12, Proposition 1.5 and its proof and Proposition 7.5]).

Now we consider the general case, and suppose that \bar{G} has a bridge. Let G_1 and G_2 be the connected components of $G \setminus \{e\}$, and set $\bar{G}_i = (G_i, \omega|_{V(G_i)})$ for $i = 1, 2$. Then we find that \bar{G}_i is a hyperelliptic or $g(\bar{G}_i) \leq 1$ (see [6, §5.2] or [12, Lemma 3.4]).

By the induction on the the number of bridges, we may and do assume that \bar{G}_i satisfies the condition (FS) for each $i = 1, 2$. Thus there exists a nodal curve X_i such that, for any $\underline{e}_i \in \text{Div}(G_i)$, there exists a Cartier divisor E_i on X_i satisfying $\rho_{i*}(E_i) = \underline{e}_i$ and $r_{X_i}(\underline{E}_i) \geq r_{\bar{G}_i}(\underline{e}_i)$, where $\rho_{i*} : \text{Div}(X_i) \rightarrow \text{Div}(G_i)$ is the map defined in (2.1).

Let p_i be a smooth point of X_i for each $i = 1, 2$. Then we patch X_1 and X_2 by $p_1 = p_2 (= p)$ to obtain a nodal curve X such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{p\}$. Here we take each p_i so that $X = X_1 \cup X_2$ is a nodal curve with dual graph \bar{G} and that each G_i is the subgraph of G corresponding to the component X_i . Let $\rho_* : \text{Div}(X) \rightarrow \text{Div}(G)$ be the map defined in (2.1).

We prove that, with this X , \bar{G} satisfies the condition (FS). Indeed, let \underline{d} be any divisor on G . For $i = 1, 2$, let \underline{d}_i be the restriction of \underline{d} to G_i . By Lemma 4.1, there exists a Cartier divisor D on X satisfying $\rho_*(D) = \underline{d}$ and

$$r_X(D) \geq \begin{cases} r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 2 & (\text{if } v_i \notin \text{Bs}(|\underline{d}_i|^\bullet) \text{ for each } i = 1, 2), \\ r_{\bar{G}_1}(\underline{d}_1 - [v_1]) + r_{\bar{G}_2}(\underline{d}_2 - [v_2]) + 1 & (\text{otherwise}). \end{cases}$$

By Lemma 4.2, the right-hand side is at least $r_{\bar{G}}(\underline{d})$. Thus we obtain $r_X(D) \geq r_{\bar{G}}(\underline{d})$, which shows that \bar{G} satisfies the condition (FS). \square

5. RANK OF DIVISORS ON GRAPHS AND CURVES OF GENUS 3

In this section, we prove Proposition 1.4 and then Theorem 1.2. Let R be a complete discrete valuation ring with fractional field \mathbb{K} and residue field k . Let \mathcal{X} be a regular, generically smooth, semi-stable R -curve. Let $\bar{G} = (G, \omega)$ be the reduction graph of \mathcal{X} . Let $\tilde{\rho}_* : \text{Div}(\mathcal{X}_{\mathbb{K}}) \rightarrow \text{Div}(G)$ be the specialization map defined in (2.2).

We begin the proof of Proposition 1.4.

Proof of Proposition 1.4. Recall that $\bar{G} = (G, \omega)$ is a non-hyperelliptic graph of genus 3 and that \mathcal{X} is a regular, generically smooth, semi-stable R -curve \mathcal{X} with reduction graph \bar{G} .

First we claim that, if $\deg(\underline{d}) \leq 2$, then $r_{\bar{G}}(\underline{d}) \leq 0$. Indeed, to argue by contradiction, suppose that $r_{\bar{G}}(\underline{d}) \geq 1$. Since $r_{\bar{G}}(\underline{d}) \leq \deg(\underline{d})$, this means (A) $\deg(\underline{d}) = 1$ and $r_{\bar{G}}(\underline{d}) = 1$; (B) $\deg(\underline{d}) = 2$ and $r_{\bar{G}}(\underline{d}) = 1$; or (C) $\deg(\underline{d}) = 2$ and $r_{\bar{G}}(\underline{d}) = 2$. In (A), the existence of \underline{d} forces \bar{G}^\bullet to be a tree, which is a contradiction. In (B), \bar{G} is hyperelliptic, which is excluded at the beginning. In (C), there exists a vertex v of \bar{G}^\bullet such that $r_{\bar{G}^\bullet}(\underline{d} - [v]) = 1$ and $\deg(\underline{d} - [v]) = 1$, where we regard $\underline{d} \in \text{Div}(\bar{G}^\bullet)$ via the natural embedding $\text{Div}(G) \subseteq \text{Div}(\bar{G}^\bullet)$ as before. The existence of the divisor $\underline{d} - [v]$ forces \bar{G}^\bullet to be a tree, which is a contradiction. Hence we obtain the claim.

Case 1. Suppose that $r_{\bar{G}}(\underline{d}) = -1$. By the surjectivity of the homomorphism (2.4), there exists $\tilde{D} \in \text{Div}(\mathcal{X}_{\mathbb{K}})$ with $\tilde{\rho}_*(\tilde{D}) = \underline{d}$. Then the specialization lemma (Theorem 2.7)

tells us that $-1 = r_{\bar{G}}(\underline{d}) \geq r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D})$. It follows that $r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D}) = -1$. Thus the equality $r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D}) = r_{\bar{G}}(\underline{d})$ holds.

Case 2. Suppose that $r_{\bar{G}}(\underline{d}) = 0$. Then it follows from Lemma 3.2 that there exists an effective divisor $\underline{e} \in \text{Div}(G)$ such that \underline{e} is linearly equivalent to \underline{d} in G . Since the homomorphism (2.4) induces a surjective map between the sets of effective divisors, there exists an effective divisor $\tilde{E} \in \text{Div}(\mathcal{X}_{\mathbb{K}})$ with $\tilde{\rho}_*(\tilde{E}) = \underline{e}$. Now we use Raynaud's theorem (Theorem A.1 below) as in the proof of [12, Theorem 1.5]. It follows that there exists a principal divisor $\tilde{N} \in \text{Div}(\mathcal{X}_{\mathbb{K}})$ such that $\tilde{\rho}_*(\tilde{N}) = \underline{d} - \underline{e}$. We set $\tilde{D} = \tilde{E} + \tilde{N}$. Then $\rho_*(\tilde{D}) = \underline{d}$. Since \tilde{D} is linearly equivalent to \tilde{E} on $\mathcal{X}_{\mathbb{K}}$, we have $r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D}) = r_{\mathcal{X}_{\mathbb{K}}}(\tilde{E}) \geq 0$. Since

$$0 = r_{\bar{G}}(\underline{d}) = r_{\bar{G}}(\underline{e}) \geq r_{\mathcal{X}_{\mathbb{K}}}(\tilde{E})$$

by the specialization lemma (Theorem 2.7), we obtain the equality $r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D}) = r_{\bar{G}}(\underline{d}) (= 0)$.

Case 3. Suppose that $r_{\bar{G}}(\underline{d}) \geq 1$. By the above claim, we have $\deg(\underline{d}) \geq 3$. We put $\underline{d}' := K_{\bar{G}} - \underline{d} \in \text{Div}(G)$. Then $\deg(\underline{d}') = 4 - \deg(\underline{d}) \leq 1$. Thus $r_{\bar{G}}(\underline{d}') \leq \deg(\underline{d}') \leq 1$. Since \bar{G}^\bullet is not a tree, we have $r_{\bar{G}}(\underline{d}') \neq 1$. It follows that $r_{\bar{G}}(\underline{d}') \leq 0$. By Cases 1 and 2, there exists a divisor $\tilde{D}' \in \text{Div}(\mathcal{X}_{\mathbb{K}})$ such that $\tilde{\rho}_*(\tilde{D}') = \underline{d}'$ and $r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D}') = r_{\bar{G}}(\underline{d}')$.

By [4, Remark 4.18 and Remark 4.21], there exists a canonical divisor $K_{\mathcal{X}_{\mathbb{K}}}$ of $\mathcal{X}_{\mathbb{K}}$ such that $\tilde{\rho}_*(K_{\mathcal{X}_{\mathbb{K}}}) = K_{\bar{G}}$. We set $\tilde{D} := K_{\mathcal{X}_{\mathbb{K}}} - \tilde{D}'$. Then we have $\tilde{\rho}_*(\tilde{D}) = K_{\bar{G}} - \underline{d}' = \underline{d}$. Further, the Riemann–Roch formulae on $\mathcal{X}_{\mathbb{K}}$ and \bar{G} (cf. [2, Theorem 3.8]) give

$$r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D}) = -2 + \deg(\tilde{D}) + r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D}') = -2 + \deg(\underline{d}) + r_{\bar{G}}(\underline{d}') = r_{\bar{G}}(\underline{d}).$$

Thus we obtain Proposition 1.4. \square

Proof of Theorem 1.2. The proof goes in the same way as in [12]; Theorem 1.2 will be deduced from Proposition 1.4.

Recall that k is a fixed algebraically closed field. We take a complete discrete valuation ring R with residue field k . For example, we may take R as the ring of formal power series $k[[t]]$ over k . Let \mathbb{K} be the fractional field of R . We take a regular, generically smooth, semi-stable R -curve \mathcal{X} with reduction graph \bar{G} . We note that such \mathcal{X} always exists: See [4, Theorem B.2].

Let $\mathcal{X}_{\mathbb{K}}$ denote the generic fiber of \mathcal{X} , and X the special fiber of \mathcal{X} . For $\underline{d} \in \text{Div}(G)$, Proposition 1.4 shows that there exists a divisor $\tilde{D} \in \text{Div}(\mathcal{X}_{\mathbb{K}})$ such that $\tilde{\rho}_*(\tilde{D}) = \underline{d}$ and $r_{\bar{G}}(\underline{d}) = r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D})$. Let $\tilde{\mathcal{D}}$ be the Zariski closure of \tilde{D} in \mathcal{X} . We denote by $\mathcal{O}_{\mathcal{X}}(\tilde{\mathcal{D}})$ the invertible sheaf on \mathcal{X} associated to $\tilde{\mathcal{D}}$. Let $D \in \text{Div}(X)$ be a divisor on X such that the associated invertible sheaf $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_{\mathcal{X}}(\tilde{\mathcal{D}})|_X$. By the upper-semicontinuity of the cohomology, we have $r_X(D) \geq r_{\mathcal{X}_{\mathbb{K}}}(\tilde{D})$. Hence $r_X(D) \geq r_{\bar{G}}(\underline{d})$. Also, by (2.3), we have $\rho_*(D) = \underline{d}$. It follows that \bar{G} satisfies the condition (FS) in Section 4, and we get $r_{\bar{G}}^{\text{alg}}(\delta) \geq r_{\bar{G}}(\delta)$ for any divisor class $\delta \in \text{Pic}(G)$. \square

In the rest of this section, we will show a metric graph version of Proposition 1.4. Let \mathcal{X} be a regular, generically smooth, semi-stable R -curve with reduction graph $\bar{G} = (G, \omega)$. Let Γ be the metric graph associated to G , where each edge of G is assigned length one. Let $\Gamma_{\mathbb{Q}}$ be the set of points of Γ whose distance from every vertex of G is rational.

We follow the arguments in [4, Section 2.3]. Let \mathbb{K}'/\mathbb{K} be a finite extension. Let R' be the ring of integers of \mathbb{K}' . Then R' is a complete discrete valuation ring with residue

field k . Let \mathcal{X}' be the minimal resolution of $\mathcal{X} \times_{\text{Spec}(R)} \text{Spec}(R')$. Then \mathcal{X}' is a regular, generically smooth, semi-stable R' -curve with generic fiber $\mathcal{X} \times_{\text{Spec}(\mathbb{K})} \text{Spec}(\mathbb{K}')$. Let $e(\mathbb{K}'/\mathbb{K})$ be the ramification index of \mathbb{K}'/\mathbb{K} . The dual graph $\bar{G}' = (G', \omega')$ of the special fiber of \mathcal{X}' is the graph obtained by inserting $e(\mathbb{K}'/\mathbb{K}) - 1$ vertices to each edge of G , and ω' is the extension of ω , where $\omega'(w) = 0$ for any $w \in V(G') \setminus V(G)$. If we assign a length of $1/e(\mathbb{K}'/\mathbb{K})$ to each edge of G' , then the corresponding metric graph equals Γ . The pair Γ with a vertex-weight function $\Gamma \rightarrow \mathbb{Z}$ (given by the zero extension of ω) is denoted by $\bar{\Gamma}$.

Let $\bar{\mathbb{K}}$ be an algebraic closure of \mathbb{K} . For $\tilde{D} \in \text{Div}(\mathcal{X}_{\bar{\mathbb{K}}})$, we take a finite extension \mathbb{K}'/\mathbb{K} such that $\tilde{D} \in \text{Div}(\mathcal{X}(\mathbb{K}'))$, and then we set $\tilde{\tau}_*(\tilde{D}) = \tilde{\rho}'_*(\tilde{D})$, where $\tilde{\rho}'_*$ is the specialization map for \mathcal{X}' . This gives rise to the specialization map

$$\tilde{\tau}_* : \text{Div}(\mathcal{X}_{\bar{\mathbb{K}}}) \rightarrow \text{Div}(\Gamma_{\mathbb{Q}}).$$

(This map is denoted by τ_* in [12]. Here we write $\tilde{\tau}_*$ instead because of the compatibility with the notation $\tilde{\rho}_*$; cf. Remark 2.6.)

For each $\underline{d} \in \text{Div}(\Gamma_{\mathbb{Q}})$, we take a graph $\bar{G}' = (G', \omega')$ with $\text{Supp}(\underline{d}) \subset V(G')$, and define $r_{\bar{\Gamma}}(\underline{d}) := r_{\bar{G}'}(\underline{d})$, which does not depend on the choice of \bar{G}' by [2, §1]. By Amini–Caporaso’s specialization lemma (Theorem 2.7), we have $r_{\bar{G}'}(\rho'_*(\tilde{D})) \geq r_{\mathcal{X}_{\bar{\mathbb{K}}}}(\tilde{D})$ for any $\tilde{D} \in \text{Div}(\mathcal{X}_{\bar{\mathbb{K}}})$. As is mentioned in [4, Remark 2.9], if $\tilde{D} \in \text{Div}(\mathcal{X}_{\bar{\mathbb{K}}}) \setminus \text{Div}(\mathcal{X}(\mathbb{K}'))$, then $\tilde{\rho}'_*(\tilde{D})$ and $\tilde{\tau}_*(\tilde{D})$ may be different, but $\tilde{\rho}_*(\tilde{D})$ and $\tilde{\tau}_*(\tilde{D})$ are at least linearly equivalent in G' . Thus, we have the specialization lemma for vertex-weighted metric graph: For any $\tilde{D} \in \text{Div}(\mathcal{X}_{\bar{\mathbb{K}}})$, one has

$$r_{\bar{\Gamma}}(\tau_*(\tilde{D})) \geq r_{\mathcal{X}_{\bar{\mathbb{K}}}}(\tilde{D}).$$

Also for metric graphs, we have Raynaud’s theorem, which asserts the surjectivity of the map $\tilde{\tau}_*|_{\text{Prin}(\mathcal{X}_{\bar{\mathbb{K}}})} : \text{Prin}(\mathcal{X}_{\bar{\mathbb{K}}}) \rightarrow \text{Prin}(\Gamma_{\mathbb{Q}})$ (see [4, Corollary A.9]). Then, by the same argument as in the proof of Proposition 1.4, we obtain the following proposition.

Proposition 5.1. *Let R be a complete discrete valuation ring with fractional field \mathbb{K} and residue field k . Let $\bar{G} = (G, \omega)$ be a non-hyperelliptic graph of genus 3, and Γ the metric graph associated to G , where each edge of G is assigned length one. Let \mathcal{X} be a regular, generically smooth, semi-stable R -curve with reduction graph \bar{G} . Then the following condition (C) holds.*

- (C) *For any $\underline{d} \in \text{Div}(\Gamma_{\mathbb{Q}})$, there exists a divisor $\tilde{D} \in \text{Div}(\mathcal{X}_{\bar{\mathbb{K}}})$ such that $\tilde{\tau}_*(\tilde{D}) = \underline{d}$ and $r_{\bar{\Gamma}}(\underline{d}) = r_{\mathcal{X}_{\bar{\mathbb{K}}}}(\tilde{D})$.*

Remark 5.2. Let R be a complete discrete valuation ring with fractional field \mathbb{K} and residue field k . Let $\bar{G} = (G, \omega)$ be a vertex-weighted graph. In [12], we have asked under what condition on \bar{G} there exists a regular, generically smooth, semi-stable R -curve with reduction graph \bar{G} that satisfies the conditions (F) and (C) in Proposition 1.4 and Proposition 5.1. In [12], when $\text{char}(k) \neq 2$, we have completely answered this question for hyperelliptic graphs: A hyperelliptic graph $\bar{G} = (G, \omega)$ satisfies the conditions (F) and (C) if and only if every vertex v of G has at most $2\omega(v) + 2$ “positive-type” bridges emanating from it. In this paper, we answer this question for non-hyperelliptic graphs of genus 3: Every non-hyperelliptic graph of genus 3 satisfies (F) and (C). It is then natural to ask this question for non-hyperelliptic graphs of genus 4. In this case, the arguments in the proof of Proposition 1.4 show the existence of a desired lift \tilde{D} of \underline{d} except for divisors \underline{d} with $\deg(\underline{d}) = 3$ and $r_{\bar{G}}(\underline{d}) = 1$.

Remark 5.3. Assume that $\text{char}(k) \neq 2$. Let $\bar{G} = (G, \omega)$ be a vertex-weighted graph. Let e_1, \dots, e_r be the set of bridges of G , and we write $G \setminus \{e_1, \dots, e_r\} = G_1 \cup \dots \cup G_{r+1}$ as the disjoint union of connected finite graphs. We set $\bar{G}_i = (G_i, \omega|_{G_i})$. The proofs of Theorem 1.1 and Theorem 1.2 show that hyperelliptic graphs and graphs of genus at most 3 satisfy the condition (FS). It follows from the proof of Theorem 1.1 that if each \bar{G}_i is hyperelliptic or of genus at most 3, then \bar{G} satisfies the condition (FS), and thus we have $r_{\bar{G}}^{\text{alg}}(\delta) \geq r_{\bar{G}}(\delta)$ for any divisor class $\delta \in \text{Pic}(G)$.

APPENDIX: RAYNAUD'S THEOREM

The purpose of this appendix is to show that, for a finite graph with loops, the specialization map between principal divisors is still surjective. Our proof of the surjectivity will be given by reducing to the case of loopless finite graphs. The surjectivity in the loopless case is shown in Baker [4].

In [4], the surjectivity of the specialization map (in the loopless case) is attributed to Raynaud because this surjectivity follows from re-interpretation of Raynaud's results in [14] (see [4, Appendix A]). In this paper, we also call Theorem A.1, which asserts the surjectivity, Raynaud's theorem.

Let k be an algebraically closed field as before. Let R be a complete valuation ring with residue field k . Let \mathbb{K} be the fractional field of R . Let $\mathcal{X} \rightarrow \text{Spec}(R)$ be a regular, generically smooth, semi-stable R -curve. We write X for the special fiber of \mathcal{X} , and $\bar{G} = (G, \omega)$ for the dual graph of X . Let $\tilde{\rho}_* : \text{Div}(\mathcal{X}_{\mathbb{K}}) \rightarrow \text{Div}(G)$ be the specialization map defined in (2.2).

Theorem A.1. *The specialization map between principal divisors is surjective. Namely, $\tilde{\rho}_*|_{\text{Prin}(\mathcal{X}_{\mathbb{K}})} : \text{Prin}(\mathcal{X}_{\mathbb{K}}) \rightarrow \text{Prin}(G)$ is surjective.*

Proof. We put $p := \text{char}(k) \geq 0$. When G is loopless, then the assertion is exactly [4, Corollary A.8]. We will reduce the general case to the loopless case.

Let d be an integer with $d \geq 2$. When $p > 0$, we require that $(d, p) = 1$. We fix a finite Galois extension \mathbb{K}' of \mathbb{K} of degree d . (For example, we may take $\mathbb{K}' = \mathbb{K}(\sqrt[d]{\pi})$, where $\pi \in R$ is a uniformizer of R .) Since k is algebraically closed and \mathbb{K}'/\mathbb{K} is a Galois extension of degree d , the ramification index $e(\mathbb{K}'/\mathbb{K})$ equals d . We denote by R' the ring of integers of \mathbb{K}' .

Let \mathcal{X}' be the minimal resolution of $\mathcal{X} \times_{\text{Spec}(R)} \text{Spec}(R')$. Let $\nu : \mathcal{X}' \rightarrow \mathcal{X}$ be the natural map. By slight abuse of notation, we denote the restriction of ν to the generic fibers by the same notation ν . Let X' be the special fiber of \mathcal{X}' . Let G' be dual graph of X' , and let $\tilde{\rho}'_* : \text{Div}(\mathcal{X}'_{\mathbb{K}'}) \rightarrow \text{Div}(G')$ be the specialization map with respect to \mathcal{X}' . Since G' is the graph obtained by inserting $(d-1)$ vertices to each edge of G , we have a natural embedding $V(G) \subseteq V(G')$ and also $\text{Div}(X) \subseteq \text{Div}(X')$.

Claim A.1.1. *Let $\tilde{D} \in \text{Div}(\mathcal{X}_{\mathbb{K}})$ such that $\tilde{\rho}'_*(\nu^*(\tilde{D})) \in \text{Div}(G)$. Then $\tilde{\rho}'_*(\nu^*(\tilde{D})) = \tilde{\rho}_*(\tilde{D})$.*

Indeed, we take any $v \in V(G)$. Let C_v be the irreducible component of X corresponding to v and let C'_v be the irreducible component of X' with $\nu(C'_v) = C_v$. Let $\mathcal{D} \in \text{Div}(\mathcal{X})$ and $\mathcal{D}' \in \text{Div}(\mathcal{X}')$ be the Zariski closures of \tilde{D} and $\nu^*(\tilde{D})$ respectively. We have $\tilde{\rho}_*(\tilde{D})(v) = (C_v \cdot \mathcal{D})$ and $\tilde{\rho}'_*(\nu^*(\tilde{D}))(v) = (C'_v \cdot \mathcal{D}')$.

Since $\nu_*(\mathcal{D}') = d\mathcal{D}$, we have $d(C_v \cdot \mathcal{D}') = (\nu^*(C_v) \cdot \mathcal{D}')$ by the projection formula. By the assumption that $\tilde{\rho}'_*(\nu^*(\tilde{D})) \in \text{Div}(G)$, we have $(E' \cdot \mathcal{D}') = 0$ for any exceptional prime

divisor E' for ν . Since $\nu^*(C_v) - dC'_v$ is a linear combination of exceptional divisors, it follows that $(\nu^*(C_v) \cdot \mathcal{D}') = d(C'_v \cdot \mathcal{D}')$.

Then

$$\tilde{\rho}_*(\tilde{D})(v) = (C_v \cdot \mathcal{D}) = \frac{(\nu^*(C_v) \cdot \mathcal{D}')}{d} = (C'_v \cdot \mathcal{D}') = \tilde{\rho}'(\nu^*(\tilde{D}))(v).$$

Since $\tilde{\rho}'_*(\nu^*(\tilde{D})) \in \text{Div}(G)$ and $v \in V(G)$ is arbitrary, we obtain Claim A.1.1.

Let $\sigma_1, \dots, \sigma_d$ be the elements of $\text{Gal}(\mathbb{K}'/\mathbb{K})$. Each σ_i induces an automorphism $\sigma_i^* : \text{Spec}(R') \rightarrow \text{Spec}(R')$, and an automorphism $\varphi_i : \mathcal{X}' \rightarrow \mathcal{X}'$ over R (induced from the cartesian product). Let $\varphi_i^* : \text{Div}(\mathcal{X}') \rightarrow \text{Div}(\mathcal{X}')$ and $\varphi_i^* : \text{Div}(\mathcal{X}_{\mathbb{K}'}') \rightarrow \text{Div}(\mathcal{X}_{\mathbb{K}'}')$ be the induced maps.

Claim A.1.2. *For any $\tilde{D}' \in \text{Div}(\mathcal{X}_{\mathbb{K}'}')$, we have $\tilde{\rho}'_*((\varphi_i)^*(\tilde{D}')) = \tilde{\rho}'_*(\tilde{D}')$ for $i = 1, \dots, d$.*

Indeed, since σ_i induces the trivial action on the residue field k , the restriction of φ_i to the special fiber X' is trivial. Thus $(\varphi_i)_*(C') = C'$ for any irreducible component C' of X' .

We take any $\tilde{D}' \in \text{Div}(\mathcal{X}_{\mathbb{K}'}')$ and let \mathcal{D}' be the Zariski closure of \tilde{D}' in \mathcal{X}' . Note that $\varphi_i^*(\mathcal{D}')$ is the Zariski closure of $\varphi_i^*(\tilde{D}')$. For any $v \in V(G')$, let C'_v be the corresponding irreducible component of X' . Then

$$\tilde{\rho}'_*(\varphi_i^*(\tilde{D}'))(v) = (C'_v \cdot \varphi_i^*(\mathcal{D}')) = ((\varphi_i)_*(C'_v) \cdot \mathcal{D}') = (C'_v \cdot \mathcal{D}') = \tilde{\rho}'_*(\tilde{D}')(v),$$

which shows the desired equality. We obtain Claim A.1.2.

We take any $\underline{n} \in \text{Prin}(G)$. Then $\underline{n} \in \text{Prin}(G')$. Since G' is loopless, we know that $\tilde{\rho}'_* : \text{Prin}(\mathcal{X}_{\mathbb{K}'}') \rightarrow \text{Prin}(G')$ is surjective by [4, Corollary A.8]. Let f be a non-zero rational function on $\mathcal{X}_{\mathbb{K}'}'$ such that $\tilde{\rho}'_*(\text{div}(f)) = \underline{n}$. We set $g' := \varphi_1^*(f) \cdots \varphi_d^*(f)$, which is a non-zero rational function on $\mathcal{X}_{\mathbb{K}'}'$. Then $\text{div}(g') = \varphi_1^*(\text{div}(f)) + \cdots + \varphi_d^*(\text{div}(f))$, so that Claim A.1.2 tells us that $\tilde{\rho}'_*(\text{div}(g')) = d\underline{n}$. Since g' is a $\text{Gal}(\mathbb{K}'/\mathbb{K})$ -invariant function on $\mathcal{X}_{\mathbb{K}'}'$, it descends to a function g on $\mathcal{X}_{\mathbb{K}}$. We have $\text{div}(g') = \nu^*(\text{div}(g))$, and thus $\tilde{\rho}'_*(\nu^*(\text{div}(g))) = d\underline{n} \in \text{Div}(G)$. By Claim A.1.1, we obtain $\tilde{\rho}_*(\text{div}(g)) = d\underline{n}$. In conclusion, $\tilde{L} := \text{div}(g)$ is a principal divisor on $\mathcal{X}_{\mathbb{K}}$ with $\tilde{\rho}_*(\tilde{L}) = d\underline{n}$.

Let $e > 2$ be another integer with $(e, d) = 1$. When $p > 0$, we require that $(e, p) = 1$. By the above argument with e in place of d , there exists a principal divisor $\tilde{M} \in \text{Prin}(\mathcal{X}_{\mathbb{K}})$ with $\tilde{\rho}_*(\tilde{M}) = e\underline{n}$. We take integers α and β such that $\alpha d + \beta e = 1$, and set $\tilde{N} := \alpha \tilde{L} + \beta \tilde{M}$. Then $\tilde{N} \in \text{Prin}(\mathcal{X}_{\mathbb{K}})$ and $\tilde{\rho}_*(\tilde{N}) = \underline{n}$. This shows the theorem. \square

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